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Statistical methods for the analysis of maximum and minimum scores in repeated measures

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ABSTRACT

A researcher may have a set of sample data that may be dependent on time or space. The research interest may not be on the comparison of the mean or median but the modal or minimum scores in the series of experiments or trials. These problems may arise in situations such as in the economy where interest is in studying transactions at the stock exchange or fluctuations in the exchange rate of money where interest is in comparing the peaks, maximum or highest; trough, minimum or lowest scores. In education and public affairs, interest may be in studying the performance of subjects or candidates in a job interview overtime or space to determine whether the subjects or candidates are equally likely to earn the highest or lowest score or grades at each of the several time points or locations. This paper proposes a non-parametric statistical method for the comparison of modes or troughs of populations matched in time or space that do not require any distributional assumptions. Sample data are used to illustrate the proposed method. The method is compared with existing Statistical methods and shown to compares favorably with the method.

Keywords: C-matched, maximum scores, minimum scores, treatment, observations, Chi-square, Cochran Q test.

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INTRODUCTION

A researcher may have a set of sample data that may be dependent on time or space. The research interest may not be on the comparison of the mean or median but the modal or minimum scores in the series of experiments or trials (Gibbons, 1971). These problems may arise in situations such as in the economy where interest is in studying transactions at the stock exchange or fluctuations in the exchange rate of money where interest is in comparing the peaks, maximum or highest; troughs, minimum or lowest scores. In education and public affairs, interest may be in studying the performance of subjects or candidates in a job interview overtime or space to determine whether the subjects or candidates are equally likely to earn the highest or lowest score or grades at each of the several time points or locations they might have taken the examinations. In public health, interest may be to assess the response of patients to a regimen of treatments or to the same treatment at different points in time, locations or hospitals and interest is in determining whether patients are equally likely to get worse or better at each of these occasions. In games or sports, the researcher may wish to statistically compare the maximum or minimum scores of players or teams over seasons or locations, etc.

In each of these and similar situations, the researcher may not properly and validly use any of the parametric tests usually employed in the analysis of matched samples because of the often intractable problems associated with determining the distribution of the modes and troughs of population distributions (Oyeka, 2009; Oyeka et al., 2012).

We therefore propose to develop a nonparametric statistical method for the comparison of modes or troughs of populations matched in time or space that do not require any distribution assumptions.

THE PROPOSED METHOD

Let x_{ij} be the i^{th} sample, block or batch of observations

randomly drawn from population x_j, for $i = 1,2,...,n;\ j = 1,2,...,c\ (c \geq 2)$. It is assumed that the c-sampled populations or treatment levels are related either in time or space and may be measurements on as low as the ordinal scale. They also need not be continuous.

To develop a test statistic that the maximum (minimum) score or observation is as likely to occur in any one treatment level as in another, we let

$$u_{ij} = \begin{cases} 1, \textit{if } x_{ij} \textit{ is the highest } (best, l\arg est) \textit{ score in sample, row or block } i \textit{ and treatment level } j \\ 0, \textit{otherwise.} \end{cases} \quad .....(1)$$

For $i = 1,2,...n,\ j = 1,2,...,c$

Note that Equation 1 may also be used to test similar null hypothesis about minimum scores if x_{ij} is redefined as the lowest (worst, smaller) score in the ith batch or block or by the ith subject at the jth treatment level, time period or location for i = 1, 2…, n; j = 1, 2,...c. Note also that $u_{ij} = 1$,for all treatment levels 'j' in which the maximum score or observation occurs for each subject 'I'. i = 1, 2,…, n; j = 1, 2,…, c. Cochran Q test may be used to test the null hypothesis by first coding all maximum (or minimum) scores 1 in each treatment level j and other observations 0 for the 'I' block or subject, i = 1, 2,.., n; j = 1, 2,…, c; that is by using the result of Equation 1 (Gibbons, 1971; Spiegel, 1998). We will however propose here, and develop an alternative approach for the same purpose.

Let $\pi_j = p(u_{ij} = 1)$.....(2)

Define

$$W_j = \sum_{i=1}^{n} u_{ij} \quad .....(3)$$

And

$$W = \sum_{j=1}^{c} W_j = \sum_{j=1}^{c}\sum_{i=1}^{n} u_{ij} \quad .....(4)$$

A null hypothesis that is usually of general interest is that each of the c treatment levels is on the average equally as likely to contain the highest (best, largest) score or observation as any other treatment level for all blocks (Gibbons, 1971; Oyeka et al., 2012). In other words, the null hypothesis of interest would be:

$H_0 : \pi_2 = ... = \pi_c = \pi$ *versus*

$$H_1 : \pi_j \neq \pi_l \quad .....(5)$$
$$j, l = 1, 2, ...c;\ j \neq l$$

$$E(u_{ij}) = \pi_j;\ \text{var}(u_{ij}) = \pi_j(1-\pi_j) \quad .....(6)$$

Equation 6 shows the expected value and variance of W_j respectively.

$$E(W_j) = \sum_{i=1}^{n} E(u_{ij}) = n\pi_j; Var(W_j) = \sum_{i=1}^{n} \text{var}(u_{ij}) = n.\pi_j(1-\pi_j) \quad .....(7)$$

Also the expected value and variance of W are respectively

$$E(W) = \sum_{j=1}^{c} E(W_j) = n\sum_{j=1}^{c} \pi_j; Var(W) = \sum_{j=1}^{c} \text{var}(W_j) = n.\sum_{j=1}^{c} \pi_j(1-\pi_j) \quad .....(8)$$

A test statistic for the null hypothesis of Equation 5 could be developed based on W of Equation 4 by finding the sampling distribution of W using Equations 1 to 4 and 6 to 8. This procedure is however rather tedious and cumbersome. We will here adopt an alternative approach based on the chi-square test for independence. Now note that π_j is the probability that on the average the highest (best, largest) observation or score occurs at the jth treatment level, time period or location for all rows or blocks of subjects for j = 1, 2,…,c. Its sample estimate is

Table 1. 2xC table for analysis of C matched sample data treatment level.

Score maximum (minimum value); 1	1	2	-	C	Total
	u_1	u_2	-	u_c	u
Score not maximum (minimum value); 0	n - u_1	n - u_2	-	n - u_c	n_c - u
Total	n	n	-	N	n_c
Proportion (p_j)	p_1	p_2	-	p_c	$\overline{p}$

$$p_j = \hat{\pi}_j = \frac{w_j}{n} = \frac{f_j}{n} \quad \text{.....(9)}$$

Where f_j is the total number of 1s in $u_{ij,}$ that is the total number of times the highest (best, largest) observation or score occurs at the jth treatment level, time period or location; for j = 1, 2,…c, for all i = 1, 2,…n, that is for all subjects or blocks of subjects. Now the overall or total number of 1s, that is the total number of highest (best, largest) scores or observations for all treatment levels, time periods or locations is

$$E(W) = \sum_{j=1}^{c} E(W_j) = n\sum_{j=1}^{c} \pi_j \quad 10$$

and

$$Var(W) = \sum_{j=1}^{c} \text{var}(w_j) = n\sum_{j=1}^{c} \pi_j (1-\pi_j) \quad \text{.....(11)}$$

The null hypothesis of Equation 5 may be tested by determining the sampling distribution of W using Equations 1 to 4 and 6 to 11. This procedure is however rather tedious and cumbersome because of the often difficult problem of determining the sampling distribution of maximum (or minimum) scores or observations in the population. We will here adopt an alternative method based on the chi-square test of independence.

Now as above, let f_j be the total number of 1s in treatment j, that is, the number of times the maximum (minimum) scores or observations by all the subjects are in treatment level j. Thus the observed number of times or frequency the maximum (minimum) scores or observations are in treatment 'j' is:

$$o_{ij} = u_j = f_j \quad \text{.....(12)}$$

So that the observed frequency of 0's, that is the total number of times, the maximum (minimum) scores for all the 'n' rows or blocks of subjects are not in treatment level j is:

$$o_{2j} = n - u_j = n - f_j \quad \text{.....(13)}$$

The corresponding proportions are:

$$p_j = \frac{u_j}{n} = \frac{f_j}{n} \quad \text{..... ....(14)}$$

and

$$q_j = 1 - p_j \quad \text{.....(15)}$$

The overall number of 1's for all the C samples is:

$$n_1 = u = f = \sum_{j=1}^{c} u_j = \sum_{j=1}^{c} f_j \quad \text{.....(16)}$$

The total number of 0's is therefore:

$$n_2 = n_c - u = n_c - f \quad \text{.....(17)}$$

The corresponding overall proportions are:

$$\overline{p} = \frac{u}{n_c} = \frac{f}{n_c} \quad \text{.....(18)}$$

and

$$\overline{q} = 1 - \overline{p} \quad \text{.....(19)}$$

These results may be summarized in a 2xC table (Table 1) for use in hypothesis testing.

Under the null hypothesis of independence, the expected frequencies corresponding to the observed ones are respectively:

$$E_{1j} = \frac{n.u}{n.c} = \frac{u}{c} \quad \text{.....(20)}$$

and

$$E_{2j} = \frac{n(n_c - u)}{n.c} = \frac{n_c - u}{c} \quad \text{.....(21)}$$

Now, under the null hypothesis H_0 of independence of all

the treatment levels, the test statistics

$$\chi^2 = \sum_{j=1}^{c}\sum_{i=1}^{2}\frac{(o_{ij} - E_{ij})^2}{E_{ij}} \quad \text{.....(22)}$$

has approximately chi-square distribution with C-1 degrees of freedom for sufficiently large 'n'. Using Equations 16, 17, 20 and 21 in Equation 22 yields

$$\chi^2 = \frac{\sum_{j=1}^{c}(\frac{u_j - u/c}{u/c})^2 + \sum_{j=1}^{c}[(n-u_j) - (nc-u)/c]^2}{nc - u/c}$$

Which when further simplified and evaluated becomes

$$\chi^2 = \frac{n}{u(nc-u)}\sum_{j=1}^{c}(c.u_j - u)^2 \quad \text{.....(23)}$$

which has approximately the chi-square distribution with C-1 degrees of freedom for sufficiently large n. The null hypothesis H_0 is rejected at the α level of significance if

$$\chi^2 \geq \chi^2_{1-\alpha,c-1} \text{ .....(24)}$$

Otherwise, H_0 is accepted.

The test statistic of Equation 24 may be alternatively expressed using the sample proportions in Equations 14, 15, 18 and 19 or in Table 1 as

$$\chi^2 = \frac{n.\sum_{j=1}^{c}(p_j - \bar{p})^2}{\bar{p}.\bar{q}} = \frac{n(\sum_{j=1}^{c}p_j^2 - c.\bar{p}^2)}{\bar{p}.\bar{q}} \quad \text{.....(25)}$$

ILLUSTRATIVE EXAMPLE

Examples are given is Tables 2 and 3.

Solution to the problem using method 2

	1	2	3	4	5	Total
$\hat{\pi}$	0.333	0.333	0.133	0.267	0.133	0.240

$$\hat{\pi} = \frac{1.199}{5} = 0.240$$

Note that for c = 2, Equation 25 reduces to

$$\chi^2 = \frac{n(p_1 - p_2)^2}{2\bar{p}\bar{q}} \quad \text{.....(26)}$$

With 1 degree of freedom where $\bar{p} = \frac{p_1 + p_2}{2}$; $\bar{q} = 1 - \bar{p}$

The test statistics of Equations 23 and 25 can, in situation where the null hypothesis of Equation 5 is rejected, be partitioned to help determine which of the sampled treatment level or levels may have contributed to the rejection of the null hypothesis. To do this, we as usual temporarily omit the treatment level that has contributed the largest to the calculated chi-square value and re-analyze the remaining treatment levels once more using the proposed method to check for the existence of any significant difference between the remaining treatment levels. This process is continued and repeated until no significant difference is found to exist between the currently remaining treatment levels (Spiegel, 1998). The remaining but not significantly different treatment levels are then pooled together. This pooled group is then compared statistically with the other treatment levels that have been found to be significantly different to guide final conclusions. At each stage during the process, it is recommended that all significant tests be conducted using critical chi-square values with C-1 degree of freedom to avoid committing Type II Error too frequently. Furthermore, when the null hypothesis of Equation 5 is rejected, it is possible to group the treatment levels into a number of mutually exclusive clusters of similar members or treatments. Then, one can compare the differential effects of these clusters by passing further analysis on the mean, median, or maximum (minimum) scores of the clusters for each of the 'n' rows or blocks of subjects (Oyeka, 2009). No new problems should arise.

Table 2. Judgment rating of 5 judges in an examination.

S/N	Judge 1	Judge 2	Judge 3	Judge 4	Judge 5
1	3	4	6	9	3
2	3	1	10	9	8
3	3	2	10	9	8
4	6	4	9	1	9
5	3	7	8	2	8
6	5	5	3	4	2
7	1	4	9	3	8
8	7	3	2	7	5
9	8	2	6	4	5
10	3	8	2	2	8
11	2	2	3	3	7
12	10	3	5	9	4
13	5	6	7	7	10
14	9	5	10	3	7
15	2	6	9	5	8

Table 3. Coding of rating of judges.

S/N	Judge 1	Judge 2	Judge 3	Judge 4	Judge 5	Total
1	1	0	0	0	1	2
2	0	1	0	0	0	1
3	0	1	0	0	0	1
4	0	0	0	1	0	1
5	0	0	0	1	0	1
6	0	0	0	0	1	1
7	1	0	0	0	0	1
8	0	0	1	0	0	1
9	0	1	0	0	0	1
10	0	0	1	1	0	2
11	1	1	0	0	0	1
12	0	1	0	0	0	1
13	1	0	0	0	0	1
14	0	0	0	1	0	1
15	1	0	0	0	0	1
Total	5	5	2	4	2	18

$$\therefore \chi^2 = \frac{(18)^2}{75(0.24)(0.76)} = \frac{324}{13.68} = 23.684$$

OR

$$\chi^2 = {(18-15)^2}\Big/{75(0.24)(0.76)} = \frac{3^2}{13.68} = \frac{9}{13.68} = 0.658 \quad (\chi^2_{0.95.4} = 0.9488)$$

Alternatively, using Cochrans Q test

$$Q = \frac{4\left(25+25+4+16+4-{(18)^2}/{5}\right)}{18-{24}/{5}} = \frac{4(74-64.8)}{13.2} = \frac{36.8}{13.2} = 2.788$$

Which with 4 degrees of freedom is not statistically significant.

Illustrative example for method 1

$$\chi^2 = \frac{n(\sum_{j=1}^{c} p_j{}^2 - c.\bar{p}^2)}{\bar{p}.\bar{q}} = \frac{15(0.333^2+0.333^2+0.133^2+0.267^2+0.133^2-5(0.240)^2)}{(0.24)(0.76)}$$

$$\chi^2 = \frac{15(0.111+0.111+0.018+0.071+0.018-(0.288))}{0.182} = \frac{15(0.329-0.288}{0.182} = \frac{15(0.041)}{0.182} = 3.379$$

which with 4 degree of freedom is not statistically significant.

SUMMARY AND CONCLUSION

We have in this paper proposed and developed a non-parametric statistical method for the analysis of maximum (or minimum) scores by subjects exposed to a battery of tests over time or space. The proposed method may be used for analyzing data measured on as low as the ordinal scale that are not necessarily continuous or numeric. A chi-square test statistic is developed to test the null hypothesis that subjects are on the average equally likely to perform or earn the highest (best, largest) scores under various treatment levels, time period, conditions or locations. In the event that the null hypothesis is rejected, the proposed method may also be used to identify the treatment level or treatment levels that may have accounted for the rejection of the null hypothesis. The proposed method is illustrated with some sample data and shown to compare favorably with the Cochran Q test.

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